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Self-Duality and Serial Rings

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Well-known theorems of Azumaya [3] and Morita [11] give necessary and sufficient conditions to insure that the category of finitely generated left modules over a (necessarily left artinian) ring is dual to the category of finitely generated right modules over a ring S. However, there is surprisingly little information available on which artinian rings (in addition to artin algebras and QF rings) have self-duality, i.e., Morita duality between their categories of finitely generated left and right modules. The categories of modules over Nakayama's serial rings are better known than those over any other significant class of non-semisimple rings. Indeed, a *serial ring* is one over which each module is a direct sum of *uniserial* modules (i.e., modules with chains for submodule lattices) that are factors of principal one-sided ideals generated by primitive idempotents (see [4, 19]). In this paper we address the question of whether serial rings have self-duality. We show that there is a large class of serial rings that do have self-duality (and we remark here that we know of none that do not).

As we have just suggested, the existing literature on rings with self-duality is sparse. The class of these rings has however been shown to include commutative rings which have a Morita duality with any ring [12, 22], linearly compact algebras over a commutative ring having self-duality [22], certain factors of skew polynomial rings [20], hereditary artinian tensor rings satisfying duality conditions [2], and artinian rings with quivers that are trees [8].

In the first section, we review the structures of an indecomposable serial ring and its minimal injective cogenerator as they are determined by a certain series of primitive left ideals, first studied by Kupisch [10] and later by Murase [16, 17, 18] and Fuller [5]. Associated with the Kupisch series is a sequence of the composition lengths of the primitive left ideals, called the admissible sequence. The second section is devoted to establishing that a serial ring with a strictly increasing admissible sequence has self-duality. This result was already known in the special case of an indecomposable hereditary serial ring, that is, a block upper triangular matrix ring over a division ring [8].

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A self-duality D over a ring R is called weakly symmetric if $D(Re/Je) \cong eR/eJ$ for every primitive idempotent $e \in R$ and $J = \operatorname{rad} R$. (Weakly symmetric self-dualities have figured importantly in W. Müller's studies of rings of finite representation type [14, 15].) The ${}_R R_R$-dual provides a self-duality if R is QF, or equivalently for an indecomposable serial ring, if the admissible sequence of R is constant [10, 16]. This duality is weakly symmetric just in case $Re/Je \cong \operatorname{Soc}(Re)$ for each primitive idempotent $e \in R$. In section three, we give a characterization of a ring with a weakly symmetric duality and show that every QF serial ring does have a weakly symmetric duality. In section four, we prove that if the lattice of submodules of every indecomposable left or right projective module over an artinian ring R with a weakly symmetric duality is a distributive lattice, then every factor ring of R has a (weakly symmetric) self-duality. Since every self-duality over a serial ring with a strictly increasing admissible sequence is weakly symmetric, we thus obtain our summarizing theorem: *Any factor ring of a serial ring with a strictly increasing or a constant admissible sequence has self-duality*. We conclude with an example of a serial ring which has self-duality but is not such a factor ring.

1. Preliminaries

Homomorphisms between left modules will be written on the right, so that fg is first f, then g; similarly, the endomorphism ring of a left module will act on the right. The natural isomorphism $\operatorname{Hom}_R(Re, Rf) \cong eRf$ will often be regarded as an identification for idempotents $e, f \in R$. If M is a left R-module, the injective envelope of M is denoted $E({}_R M)$ and the composition length $c({}_R M)$ The right annihilator of X in Y is $r_Y(X) = \{y \in Y \mid Xy = 0\}$; in particular, if $J = \operatorname{rad} R$, then $\operatorname{Soc}_k(M) = r_M(J^k)$.

An artinian ring R is said to have a *self-(Morita) duality* if there is a Morita duality D between R-mod, the category of finitely generated left R-modules, and mod-R, the category of finitely generated right R-modules. Since we are assuming that R is artinian, Morita [11] and Azumaya [3] have shown:

(1.1) *R has a self-duality D if and only if there is an injective cogenerator ${}_R E$ of* R-mod *and a ring isomorphism φ: $R \to \operatorname{End}({}_R E)$ (which induces a right R-structure on E via $x \cdot r = x\varphi(r)$ for $x \in E$ and $r \in R$), such that the dualities D and* $\operatorname{Hom}_R(\ , {}_R E_R)$ *are naturally equivalent.*

Assume now that R is an indecomposable serial ring with $J = \operatorname{rad} R$. Let $\{e_1, \ldots, e_n\}$ be a basic set of primitive idempotents of R and let $c_i = c(Re_i)$ for $i = 1, \ldots, n$. Then the unique composition series of Re_i is $Re_i = J^0 e_i \supseteq Je_i \supseteq J^2 e_i \supseteq \cdots \supseteq J^{c_i} e_i = 0$, and $\operatorname{Soc}_k(Re_i) = J^{c_i - k} e_i$. For each integer j, let $[j]$ denote the least strictly positive remainder of j modulo n; in particular, $[jn] = n$. Then the basic set of primitive idempotents can be indexed so that [10]:

(1.2) (1) *$Re_i/Je_i \cong Je_{i+1}/J^2e_{i+1}$ ($i = 1,..., n - 1$) and $Re_n/Je_n \cong Je_1/J^2e_1$ if $Je_1 \neq 0$;*

(2) *$c_i \geqslant 2$ ($i = 2,..., n$); and*

(3) *$c_{[i+1]} \leqslant c_i + 1$ ($i = 1,..., n$).*

We will also insist that the indexing be chosen to insure:

(4) *$c_1 \leqslant c_i$ ($i = 1,..., n$).*

The series $Re_1 ,..., Re_n$ is called a *(left) Kupisch series* of R and $c_1 ,..., c_n$ is the corresponding *admissible sequence* of R. From property (1), it quickly follows that [5, Lemma 2.1]:

(1.3) *If $J^k e_j \neq 0$ then $J^k e_j \cong Re_{[j-k]}/J^{c_j-k}e_{[j-k]}$.*

Murase [18] defines a *chain end* of R to be a member Re_i of the Kupisch series with $c_{[i+1]} \leqslant c_i$. The chain ends are precisely the indecomposable injective projective left R-modules, and every indecomposable injective left R-module is an epimorph of a chain end [5, Theorem 2.5]:

(1.4) *For each i, $Re_{[i+b_i-1]}$ is a chain end and $E(Re_i/Je_i) \cong Re_{[i+b_i-1]}/J^{b_i}e_{[i+b_i-1]}$, where $b_i = c(e_iR_R)$.*

Every indecomposable module M over a serial ring R is both quasi-injective and quasi-projective; that is, for every submodule $N \subseteq M$, every homomorphism $N \to M$ extends to an endomorphism of M, and every homomorphism $M \to M/N$ factors through the natural epimorphism [5, Theorem 5.4]. Then also for submodules $K \subseteq N \subseteq M$, every diagram

$$\begin{tikzcd} & M & \\ 0 \arrow[r] & K \arrow[u] \arrow[r] & N \arrow[ul, dashed] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & M \arrow[dl, dashed] \arrow[d] & \\ M/K \arrow[r] & M/N \arrow[r] & 0 \end{tikzcd}$$

can be completed.

A principal technique used in the proofs of (2.4) and (3.2) is that of changing the domain or the range of a function. For example, the factor theorem states that over any ring R, if $f: K \to M$ is an R-homomorphism and $g: K \to N$ is an epimorphism with $\ker g \subseteq \ker f$, then there is a homomorphism $h: N \to M$ such that $gh = f$. If M is quasi-injective, we can replace the requirement that $g: K \to N$ be an epimorphism by hypothesizing the existence of a monomor- $N/(\ker f)g \to M$. (In the factor theorem, $\bar{h}: N/(\ker f)g \to M$ defined by $\bar{h}: x + (\ker f)g \mapsto (x)h$ for $x \in N$ is a monomorphism.) For consider the commutative diagram

$$\begin{tikzcd} K \arrow[r, "\eta"] \arrow[d, "g"'] & K/\ker f \arrow[r, "\bar{f}"] \arrow[d, "\bar{g}"] & M \arrow[d, equal] \\ N \arrow[r, "\eta"] & N/(\ker f)g \arrow[r, "\bar{h}"] & M, \end{tikzcd}$$

where $\bar{g}$ is the induced monomorphism and η is the natural epimorphism. Assuming that $N/(\ker f)g$ embeds into M, the quasi-injectivity of M yields a map $\bar{h}\colon N/(\ker f)g \to M$ that makes the diagram commute, and we take $h = \eta\bar{h}\colon N \to M$. Thus we have shown:

(1.5) *Assume that M is quasi-injective and let $f\colon K \to M$ and $g\colon K \to N$ be homomorphisms such that $\ker g \subseteq \ker f$. If there exists a monomorphism $N/(\ker f)g \to M$, then there is a homomorphism $h\colon N \to M$ such that $gh = f$:*

$$\begin{tikzcd} K \arrow[r, "g"] \arrow[dr, "f"'] & N \arrow[d, dashed, "h"] \\ & M. \end{tikzcd}$$

By a dual argument one can show:

(1.6) *Assume that M is quasi-projective and let $f\colon M \to K$ and $g\colon N \to K$ be homomorphisms such that $\operatorname{im} g \supseteq \operatorname{im} f$. If there exists an epimorphism $M \to (\operatorname{im} f)\, g^{-1}$, then there is a homomorphism $h\colon M \to N$ such that $hg = f$:*

$$\begin{tikzcd} M \arrow[d, dashed, "h"'] \arrow[dr, "f"] & \\ N \arrow[r, "g"'] & K. \end{tikzcd}$$

Finally, assume R is serial, $J = \operatorname{rad} R$, and M and N are indecomposable (hence, uniserial) R-modules. If $\operatorname{Soc}(M) \cong \operatorname{Soc}(N)$, then $E(M)$ and $E(N)$ are isomorphic and also uniserial, so there is a monomorphism either from M to N or from N to M. This and a dual argument demonstrate:

(1.7) *Let M and N be indecomposable modules over a serial ring R with $c(M) \leqslant c(N)$.*

(1) *If $\operatorname{Soc}(M) \cong \operatorname{Soc}(N)$, then there is a monomorphism $M \to N$.*

(2) *If $M/JM \cong N/JN$, then there is an epimorphism $N \to M$.*

2. Serial Rings with Strictly Increasing Admissible Sequences

Now we assume that R is an indecomposable serial ring and that the admissible sequence of R is strictly increasing, that is, that $c_{i+1} = c_i + 1$ for $i = 1,\ldots, n-1$. Then from the results cited in section one, Re_n is the only indecomposable injective projective left R-module; every indecomposable projective left R-module embeds in Re_n; and every indecomposable injective left R-module is isomorphic to a factor of Re_n. Playing a role dual to that of Re_n is the module $U = J^n e_n \cong Je_1$:

(2.1) Lemma. *Let R be an indecomposable serial ring with a strictly increasing admissible sequence. Let $E_i = E(Re_i/Je_i)$ be the injective envelope of Re_i/Je_i and let $U = J^n e_n$. Let $k_i = [c_n + i]$. Then*

(1) *Each indecomposable projective module embeds in Re_n: $Re_i \cong J^{n-i}e_n = \mathrm{Soc}_{c_n-n+i}(Re_n)$;*

(2) *U embeds properly into each indecomposable projective: $U \cong J^i e_i = \mathrm{Soc}_{c_i-i}(Re_i)$;*

(3) *Each indecomposable injective is an epimorph of Re_n: $E_i \cong Re_n/J^{c_n-k_i+1}e_n = Re_n/\mathrm{Soc}_{k_i-1}(Re_n)$; and*

(4) *U is a proper epimorph of each indecomposable injective: $U \cong E_i/J^{c_n-n}E_i = E_i/\mathrm{Soc}_{n-k_i+1}(E_i)$.*

Proof. Assume that the admissible sequence of R is $c_1 < c_2 < \cdots < c_n$. By (1.3) and the hypothesis $c_n \geqslant n$, $0 \neq J^{n-i}e_n \cong Re_i/J^{c_n-(n-i)}e_i$. But $c_n = c_i + (n-i)$, so $J^{c_n-(n-i)}e_i = 0$ and the first conclusion follows. Next, since $U \cong J^n e_n$, $c(U) = c_n - n < c_i$ and $\mathrm{Soc}(U) = \mathrm{Soc}(Re_n) \cong \mathrm{Soc}(Re_i)$, so U embeds properly into each indecomposable projective. Comparing lengths gives $U \cong J^i e_i$.

For (3) and (4) recall (1.4) $E_i \cong Re_{[i+b_i-1]}/J^{b_i}e_{[i+b_i-1]}$ where $b_i = c(e_iR)$. Since Re_n is the only chain end, we must have $pn = i + b_i - 1$ for some integer p and $E_i \cong Re_n/J^{b_i}e_n$. Because $0 \leqslant c_n - b_i < n$ (see [5, Theorem 2.5]), $c_n - b_i + 1 = [c_n - b_i + 1] = [c_n + i] = k_i$ and (3) follows. Finally, since $c(U) = c_n - n < c_n - k_i + 1 = c(E_i)$ and $U/JU \cong E_i/JE_i$, U is a proper factor of E_i. Comparing composition lengths gives $U \cong E_i/J^{c_n-n}E_i = E_i/\mathrm{Soc}_{n-k_i+1}(E_i)$.

The two modules Re_n and U are fundamental to the argument of the main theorem of this section.

We shall need two other technical lemmas in the proof of Theorem ((2.4) The first analyzes maps between indecomposable projectives; the second concerns the structure of indecomposable injectives.

(2.2) Lemma. *Let R be an indecomposable serial ring with Kupisch series*

$Re_1, \ldots, Re_n$. If $1 \leqslant j < i \leqslant n$ and $g\colon Re_i \to Re_j$ is an R-homomorphism, then $\operatorname{im} g \subseteq J^{j+n-i}e_j$; moreover, if the admissible sequence of R is strictly increasing and $g \neq 0$, then $c(\ker g) \geqslant n$.

Proof. By (1.3) $e_i(J^k e_j/J^{k+1}e_j) \neq 0$ only if $[j-k] = i$. Since $j+n-i < n$, the least k with $[j-k] = i$ is $k = j+n-i$. Thus $\operatorname{im} g \subseteq J^{j+n-i}e_j$. Notice that if $g \neq 0$, then $J^{j+n-i}e_j \neq 0$ and $c(\operatorname{im} g) \leqslant c_j - (j+n-i)$. If R has a strictly increasing admissible sequence, then $c_i = c_j + (i-j)$. Hence $c(\ker g) = c_i - c(\operatorname{im} g) \geqslant c_i - (c_j - (j+n-i)) = n$.

For a left R-module M and $i \in \{1, \ldots, n\}$, define $d_i(M) = c({}_{e_iRe_i}e_iM)$ to be the number of composition factors of M isomorphic to Re_i/Je_i.

(2.3) Lemma. *Let R be an indecomposable serial ring with strictly increasing admissible sequence $c_1 < c_2 < \cdots < c_n$. Then for $i \in \{1, \ldots, n\}$,*

(1) *$d_n(Re_n) - 1 = d_n(U) \leqslant d_n(E_i) = d_i(E_i) = d_i(Re_n) \leqslant d_n(Re_n)$; and*

(2) *$c(E_i) = nd_n(E_i) - i + 1$.*

Proof. Let $S_j \cong Re_j/Je_j$. By (2.1) and (1.3), the composition factors of E_i are, in order, $(*)$ $E_i/JE_i \cong S_n, S_{n-1}, \ldots, S_{[i+1]}, S_i, S_{[i-1]}, \ldots, S_1, S_n, \ldots, S_i, \ldots, S_1, \ldots, S_n, \ldots, S_{[i+1]}, S_i \cong \operatorname{Soc}(E_i)$.

From $J^j e_n/J^{j+1}e_n \cong S_{n-j}$ for $j \in \{0, \ldots, n-1\}$, it follows that $d_n(U) = d_n(J^n e_n) = d_n(Re_n) - 1$. Because U is a factor of E_i and E_i of Re_n, $d_n(U) \leqslant d_n(E_i) \leqslant d_n(Re_n)$. From $(*)$ it is clear that $d_n(E_i) = d_i(E_i)$. That E_i is a factor of Re_n implies $d_i(E_i) \leqslant d_i(Re_n)$. Since E_i is a maximal extension of S_i and since every factor of Re_n is uniform (every submodule is essential), we must also have $d_i(E_i) \geqslant d_i(Re_n)$, and (1) is proven. For (2), note that by $(*)$, $c(E_i) = (n-i) + 1 + (d_n(E_i) - 1)n = nd_n(E_i) - i + 1$.

(2.4) Theorem. *An indecomposable serial ring with a strictly increasing admissible sequence has self-duality.*

Proof. Let R be an indecomposable serial ring with $\{e_1, \ldots, e_n\}$ a basic set of primitive idempotents of R and admissible sequence $c_1 < c_2 < \cdots < c_n$. We may assume that R is basic and that $c_1 > 1$, for if $c_1 = 1$, then R is a triangular matrix ring and the theorem follows from [8, Proposition 4]. Set $J = \operatorname{rad} R$, $U = J^n e_n$, and $E_i = E(Re_i/Je_i)$. Let $E = \bigoplus_{i=1}^n E_i$ be the minimal injective cogenerator of R-mod, and let $S = \operatorname{End}(E)$ with $f_i \in S$ the natural projection onto E_i. We will exhibit a ring isomorphism $\Phi\colon S \to R$ with $\Phi(f_i) = e_i$. An element $s \in S$ may be represented as an $(n \times n)$-matrix with the (i,j)-entry equal to $f_i s f_j$, which we regard as an R-homomorphism from E_i to E_j. The ring isomorphism Φ will first be defined as an epimorphism from $f_i S f_j$ to $e_i R e_j$ for each pair (i,j), then extended additively to a function from S to R. To this end, define α_i and β_i as follows: If $d_n(E_i) = d_n(Re_n)$, let $\alpha_i\colon Re_n \to E_i$ be an epimorphism and $\beta_i\colon Re_i \to Re_n$ a monomorphism; if $d_n(E_i) = d_n(Re_n) - 1$, let $\alpha_i\colon E_i \to U$ be an epimorphism and $\beta_i\colon U \to Re_i$ a monomorphism. The maps α_i and β_i ($i = 1,\ldots, n$) will remain fixed throughout the proof. Under each case below, we first define $\Phi(f_i s f_j) = e_i r e_j$, then verify that Φ is a well-defined function, and finally show that Φ is onto:

(i) $d_n(E_i) = d_n(Re_n) = d_n(E_j)$. Consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Re}_i \arrow[r, "\beta_i"] \arrow[d, dashed, "e_i r e_j"'] \arrow[dr, dashed, "\epsilon"] & \mathrm{Re}_n \arrow[r, "\alpha_i"] \arrow[d, dashed, "\delta"] \arrow[dr, dashed, "\gamma"] & E_i \arrow[d, "f_i s f_j"] \\
\mathrm{Re}_j \arrow[r, "\beta_j"] & \mathrm{Re}_n \arrow[r, "\alpha_j"] & E_j .
\end{tikzcd}$$

Let $\gamma = \alpha_i f_i s f_j$; then the map δ exists since α_j is an epimorphism and Re_n is projective. Let $\epsilon = \beta_i \delta$. The map $e_i r e_j$ exists if $\operatorname{im} \epsilon \subseteq \operatorname{im} \beta_j$ since β_j is a monomorphism. If $i \leqslant j$ then $\operatorname{im} \epsilon \subseteq J^{n-i} e_n \subseteq J^{n-j} e_n = \operatorname{im} \beta_j$. Assume $i > j$. From (2.3) and the hypothesis $d_n(E_i) = d_n(E_j)$, $c(E_i) = n d_n(E_i) - i + 1 < n d_n(E_j) - j + 1 = c(E_j)$. Since $E_i/JE_i \cong Re_n/Je_n$, it follows that $(\operatorname{im} \gamma)\, \alpha_j^{-1} = (\operatorname{im} f_i s f_j)\, \alpha_j^{-1} \subseteq (J^n E_j)\, \alpha_j^{-1} = J^n e_n + \ker \alpha_j$. But because $d_n(E_j) = d_n(Re_n)$, $0 = e_n \ker \alpha_j$ and $\ker \alpha_j \subseteq J^n e_n$. Hence $\operatorname{im} \epsilon \subseteq \operatorname{im} \delta \subseteq (\operatorname{im} \gamma)\, \alpha_j^{-1} \subseteq J^n e_n \subseteq J^{n-j} e_n = \operatorname{im} \beta_j$. Therefore $e_i r e_j$ does exist.

Since the relation Φ is additive on $f_i S f_j$ (to $f_i s f_j + f_i s' f_j$ correspond the maps $\gamma + \gamma'$, $\delta + \delta'$, $\epsilon + \epsilon'$, and $e_i r e_j + e_i r' e_j$), in order to demonstrate that Φ is a well-defined function, it suffices to show that if $e_i r e_j \neq 0$ and $e_i r e_j$ corresponds to $f_i s f_j$, then $f_i s f_j \neq 0$. Accordingly, assume $e_i r e_j \neq 0$. Then since β_j is a monomorphism, $0 \neq e_i r e_j \beta_j = \epsilon = \beta_i \delta$ and $\delta \neq 0$. Next, $\gamma = \delta \alpha_j = 0$ iff $\operatorname{im} \delta \subseteq \ker \alpha_j$. Now $d_n(Re_n) = d_n(E_j)$, so $e_n \ker \alpha_j = 0$, but $(e_n)\delta \neq 0$. Thus $\operatorname{im} \delta \nsubseteq \ker \alpha_j$ and $\gamma \neq 0$. Hence $\alpha_i f_i s f_j = \gamma \neq 0$, so $f_i s f_j \neq 0$ and Φ is a well-defined function.

Finally, to show that $\Phi(f_i S f_j) = e_i R e_j$, let $e_i r e_j \neq 0$ be given. Let $\epsilon = e_i r e_j \beta_j$; then the map δ exists since β_i is a monomorphism and Re_n is injective. Let $\gamma = \delta \alpha_j$. By the factor theorem, $f_i s f_j$ exists if $\ker \alpha_i \subseteq \ker \gamma$ since α_i is an epimorphism. If $i \leqslant j$, then from (2.3) and the hypothesis $d_n(E_i) = d_n(E_j)$, the inequality $c(E_i) \geqslant c(E_j)$ follows. Hence, since submodules (particular $\ker \alpha_j$) of uniserial modules are stable under endomorphisms, $c(\ker \gamma) = c(\ker \delta \alpha_j) \geqslant c(\ker \alpha_j) = c(Re_n) - c(E_j) \geqslant c(Re_n) - c(E_i) = c(\ker \alpha_i)$. If $i > j$, then by (2.2) $c(\ker e_i r e_j) \geqslant n$, so $c(\ker \gamma) \geqslant c(\ker \beta_i \gamma) = c(\ker e_i r e_j \beta_j \alpha_j) \geqslant c(\ker e_i r e_j) \geqslant n > c(\ker \alpha_i)$. Therefore $\ker \alpha_i \subseteq \ker \gamma$ and $f_i s f_j$ does exist.

(ii) $d_n(E_i) = d_n(Re_n) > d_n(U) = d_n(E_j)$. Consider the following commutative diagram:

$$\begin{tikzcd}
Re_i \arrow[r, "\beta_i"] \arrow[d, dashed, "e_i r e_j"'] & Re_n \arrow[r, "\alpha_i"] \arrow[d, dashed, "\delta"] \arrow[dl, dashed, "\epsilon"'] & E_i \arrow[d, "f_i s f_j"] \arrow[dl, dashed, "\gamma"'] \\
Re_j \arrow[r, "\beta_j"] & U \arrow[r, "\alpha_j"] & E_j
\end{tikzcd}$$

In this case, we may define γ, δ, ϵ, and e_ire_j by composition; in particular, $\Phi(f_isf_j) = \beta_i\alpha_if_isf_j\alpha_j\beta_j = e_ire_j$. That Φ is well-defined is clear.

In order to show that Φ is onto, let e_ire_j be given with $e_ire_j \neq 0$. The map ϵ exists by (1.5) if $c(Re_n/(\ker e_ire_j)\,\beta_i) \leqslant c(Re_j)$ since $\mathrm{Soc}(Re_n/(\ker e_ire_j)\,\beta_i) \cong \mathrm{Soc}(Re_i/\ker e_ire_j) \cong \mathrm{Soc}(Re_j)$. Because $d_i(Re_n) = d_n(E_i) > d_n(E_j) = d_j(Re_n)$ $i > j$ and $c(\ker e_ire_j) \geqslant n$ by (2.2). Thus $c(Re_n/\ker(e_ire_j)\,\beta_i) = c(Re_n) - c(\ker e_ire_j) \leqslant c(Re_n) - n < c(Re_j)$, and ϵ does exist. Since $\mathrm{im}\,\epsilon \subseteq J^je_j = \mathrm{im}\,\beta_j$ by (2.2) and β_j is a monomorphism, δ exists. Next, γ exists by the factor theorem, since α_i is an epimorphism and $c(\ker\alpha_i) < n = c(Re_n) - c(U) \leqslant c(\ker\delta)$. Finally, $d_n(E_i) > d_n(E_j)$ implies $c(E_i) > c(E_j) \geqslant c((\mathrm{im}\,\gamma)\,\alpha_j^{-1})$, so f_isf_j exists by (1.6) since both E_i and $(\mathrm{im}\,\gamma)\,\alpha_j^{-1}$ are factors of Re_n.

(iii) $d_n(E_i) = d_n(U) < d_n(Re_n) = d_n(E_j)$. Consider the following commutative diagram:

$$\begin{tikzcd}
Re_i \arrow[r, "\beta_i"] \arrow[d, dashed, "e_ire_i"'] \arrow[dr, dashed, "\epsilon"] & U \arrow[r, "\alpha_i"] \arrow[d, dashed, "\delta"] \arrow[dr, dashed, "\gamma"] & E_i \arrow[d, "f_isf_j"] \\
Re_j \arrow[r, "\beta_j"] & Re_n \arrow[r, "\alpha_j"] & E_j
\end{tikzcd}$$

Since α_i is an epimorphism, the map γ exists by the factor theorem if $c(\ker\alpha_i) < c(\ker f_isf_j)$. But $d_n(E_i) < d_n(E_j)$, so $\mathrm{im}\,f_isf_j \subseteq J^nE_j$, an epimorph of $U = J^ne_n$. And now $c(\ker\alpha_i) + c(U) = c(E_i) = c(\ker f_isf_j) + c(\mathrm{im}\,f_isf_j) \leqslant c(\ker f_isf_j) + c(U)$. Thus $c(\ker\alpha_i) \leqslant c(\ker f_isf_j)$ and γ does exist. Since $U\gamma \subseteq J^nE_j$, $c((U\gamma)\,\alpha_j^{-1}) \leqslant c((J^nE_j)\,\alpha_j^{-1}) = c(U)$, so δ exists by (1.6). Next, ϵ exists because β_i is a monomorphism and Re_n is injective. Finally, since $d_i(Re_n) = d_n(E_i) < d_n(E_j) = d_j(Re_n)$, $i < j$, $c_i < c_j$, and $\mathrm{im}\,\epsilon \subseteq \mathrm{im}\,\beta_j$. Thus e_ire_j exists since β_j is a monomorphism.

Now assuming that $e_ire_j \neq 0$, also $\epsilon = e_ire_j\beta_j \neq 0$ since β_j is a monomorphism. Because $1 + d_n(E_i) = d_n(Re_n) = d_n(E_n)$, (2.1) implies that $c(Re_n) - c(E_i) \geqslant c(E_n) - c(E_i) = i$, and hence, $e_i\,\mathrm{Soc}_i(Re_n) = 0$. Since $\epsilon \neq 0$, $\mathrm{im}\,\epsilon \supseteq \mathrm{Soc}_{i+1}(Re_n)$. $c(\ker\epsilon) = c(Re_i) - c(\mathrm{im}\,\epsilon) \leqslant c(Re_i) - (i+1) < c(\mathrm{im}\,\beta_i)$, and $\mathrm{im}\,\beta_i \not\subseteq \ker\epsilon$ Therefore $\delta = \beta_i\epsilon \neq 0$. Next, $\gamma = \delta\alpha_j = 0$ iff $\mathrm{im}\,\delta \subseteq \ker\alpha_j$. Now $d_n(Re_n) = d_n(E_j)$, so $e_n\ker\alpha_j = 0$, but $(e_n)\delta \neq 0$. Thus $\mathrm{im}\,\delta \not\subseteq \ker\alpha_j$ and $\gamma \neq 0$. Finally, $f_isf_j = \alpha_i\gamma \neq 0$ since α_i is an epimorphism, and Φ is a well-defined function. It is clearly onto, taking $f_isf_j = \alpha_i\beta_ie_ire_j\beta_j\alpha_j$ for $e_ire_j \in e_iRe_j$.

(iv) $d_n(E_i) = d_n(U) = d_n(E_j)$. Consider the following commutative diagram:

$$\begin{tikzcd}
Re_i \arrow[r, "\beta_i"] \arrow[d, dashed, "e_ire_j"'] & U \arrow[r, "\alpha_i"] \arrow[d, dashed, "\delta"] \arrow[dl, dashed, "\epsilon"'] & E_i \arrow[d, "f_isf_j"] \arrow[dl, dashed, "\gamma"'] \\
Re_j \arrow[r, "\beta_j"] & U \arrow[r, "\alpha_j"] & E_j
\end{tikzcd}$$

Let $\gamma = f_isf_j\alpha_j$; then δ exists by the factor theorem since $\ker \alpha_i \subseteq \ker \gamma$. Let $\epsilon = \delta\beta_j$. The map e_ire_j exists by (1.5) if $c(Re_i/(\ker \epsilon)\,\beta_i) \leqslant c(Re_j)$. If $i \leqslant j$, then $c(Re_i/(\ker \epsilon)\,\beta_i) \leqslant c(Re_i) \leqslant c(Re_j)$. If $i > j$, then (2.3) and $d_n(E_i) = d_n(E_j)$ imply $c(E_i) < c(E_j)$, so $\operatorname{im} f_isf_j \subseteq J^nE_j$ and $\operatorname{im} \delta = \operatorname{im} f_isf_j\alpha_j \subseteq J^nU$. Thus, either $\operatorname{im} \epsilon = 0$ and we take $e_ire_j = 0$, or $c(\operatorname{im} \epsilon) = c(\operatorname{im} \delta) \leqslant c(U) - n$ and thus $c(\ker \epsilon) \geqslant n$, $c(Re_j) \geqslant c(Re_n) - n \geqslant c(Re_i) - n \geqslant c(Re_i/(\ker \epsilon)\,\beta_i)$, and $\Phi(f_isf_j) = e_ire_j$ does exist.

Assume that $e_ire_j \neq 0$. Then setting $k = c(\operatorname{im} e_ire_j)$, $0 \neq e_i(\operatorname{im} e_ire_j) = e_i \operatorname{Soc}_k(Re_j) \simeq e_i \operatorname{Soc}_k(Re_n)$. Since $d_n(E_i) < d_n(Re_n)$, again we see that $k \geqslant i + 1$, and $c(\ker e_ire_j) = c_i - c(\operatorname{im} e_ire_j) < c_i - i = c(\operatorname{im} \beta_i)$. Therefore $\operatorname{im} \beta \not\subseteq \ker e_ire_j$ and $\epsilon = \beta_ie_ire_j \neq 0$. And now, since α_i is an epimorphism, $0 \neq \alpha_i\epsilon = f_isf_j\alpha_j\beta_j$, so $f_isf_j \neq 0$ and Φ is well-defined.

Finally, let e_ire_j be given and set $\epsilon = \beta_ie_ire_j$. The map δ exists since $\operatorname{im} \epsilon \subseteq \operatorname{im} \beta_j$ and β_j is a monomorphism; let $\gamma = \alpha_i\delta$. By (1.6) f_isf_j exists if $c(E_i) \geqslant c((\operatorname{im} \gamma)\,\alpha_j^{-1})$. If $i \leqslant j$, then $d_n(E_i) = d_n(E_j)$ implies $c(E_i) \geqslant c(E_j) \geqslant c((\operatorname{im} \gamma)\,\alpha_j^{-1})$. If $i > j$, then $\operatorname{im} e_ire_j \subseteq J^{j+n-i}e_j$, so $\operatorname{im} \epsilon \subseteq J^{j+n}e_j$. Hence $\operatorname{im} \gamma \subseteq J^nU$ and either $\gamma = 0$ and we take $f_isf_j = 0$, or $c((\operatorname{im} \gamma)\,\alpha_j^{-1}) = c(\operatorname{im} \gamma) + c(\ker \alpha_j) \leqslant c(U) - n + c(\ker \alpha_j) = c(E_j) - n < c(E_i)$. Therefore f_isf_j exists and Φ is onto.

Now we may define $\Phi\colon S \to R$ via $\Phi\colon s \mapsto \Sigma_{i,j}\Phi(f_isf_j)$. That Φ is surjective and additive follows from the results in (i)–(iv) above and the abelian group direct sum decompositions $R = \Sigma_{i,j}e_iRe_j$ and $S = \Sigma_{i,j}f_iSf_j$. The function Φ is multiplicative since, given f_isf_j and $f_js'f_k$, the diagram and maps γ'', δ'', ϵ'', and $e_ir''e_k$ corresponding to $f_isf_js'f_k$ are given by writing the (i,j)- and (j,k)-diagrams adjacent to each other and preceding γ', δ', ϵ' by f_isf_j, δ, or e_ire_j, as is appropriate. For example, if $d_n(E_i) = d_n(U) = d_n(E_k) < d_n(E_j) = d_n(Re_n)$, we may take $e_ir''e_k = e_ire_jr'e_k$:

$$\begin{tikzcd}
Re_i \arrow[r, "\beta_i"] \arrow[d, "e_ire_j"'] \arrow[dr] & U \arrow[r, "\alpha_i"] \arrow[d, "\delta"] \arrow[dr] & E_i \arrow[d, "f_isf_j"] \\
Re_j \arrow[r, "\beta_j"] \arrow[d, "e_jr'e_k"'] & Re_n \arrow[r, "\alpha_j"] \arrow[d, "\delta'"] \arrow[dl, "\epsilon'"'] & E_j \arrow[d, "f_js'f_k"] \arrow[dl, "\gamma'"'] \\
Re_k \arrow[r, "\beta_k"] & U \arrow[r, "\alpha_k"] & E_k
\end{tikzcd}$$

(There are 8 cases; see Fig. 1.) Thus $\Phi(s)\,\Phi(s') = (\Sigma_{i,j}e_ire_j)(\Sigma_{j,k}e_jr'e_k) = \Sigma_{i,j,k}e_ire_jr'e_k = \Phi(ss')$. Therefore Φ is a surjective ring homomorphism. Since R is basic, $c(S_S) = c(\operatorname{Hom}_S(S, {}_RE)) = c({}_RE) = c(R_R)$ by (1.4), and $\ker \Phi = 0$; hence, Φ is an isomorphism. Therefore R has self-duality.

$$\begin{tikzcd}
Re_i \arrow[r] \arrow[d] \arrow[dr] & Re_n \arrow[r] \arrow[d] \arrow[dr] & E_i \arrow[d] \\
Re_j \arrow[r] \arrow[d] \arrow[dr] & Re_n \arrow[r] \arrow[d] \arrow[dr] & E_j \arrow[d] \\
Re_k \arrow[r] & Re_n \arrow[r] & E_k
\end{tikzcd}$$

i,i,i

$$\begin{tikzcd}
Re_i \arrow[r] \arrow[d] \arrow[dr] & Re_n \arrow[r] \arrow[d] \arrow[dr] & E_i \arrow[d] \\
Re_j \arrow[r] \arrow[d] & Re_n \arrow[r] \arrow[d] \arrow[dl] & E_j \arrow[d] \arrow[dl] \\
Re_k & U \arrow[l] & E_k \arrow[l]
\end{tikzcd}$$

i,ii,ii

$$\begin{tikzcd}
Re_i \arrow[r] \arrow[d] & Re_n \arrow[r] \arrow[d] \arrow[dl] & E_i \arrow[d] \arrow[dl] \\
Re_j \arrow[d] \arrow[dr] & U \arrow[l] \arrow[d] \arrow[dr] & E_j \arrow[l] \arrow[d] \\
Re_k \arrow[r] & Re_n \arrow[r] & E_k
\end{tikzcd}$$

ii,iii,i

$$\begin{tikzcd}
Re_i \arrow[r] \arrow[d] & Re_n \arrow[r] \arrow[d] \arrow[dl] & E_i \arrow[d] \arrow[dl] \\
Re_j \arrow[d] & U \arrow[l] \arrow[d] \arrow[dl] & E_j \arrow[l] \arrow[d] \arrow[dl] \\
Re_k & U \arrow[l] & E_k \arrow[l]
\end{tikzcd}$$

ii,iv,ii

$$\begin{tikzcd}
Re_i \arrow[d] \arrow[dr] & U \arrow[l] \arrow[d] \arrow[dr] & E_i \arrow[l] \arrow[d] \\
Re_j \arrow[r] \arrow[d] \arrow[dr] & Re_n \arrow[r] \arrow[d] \arrow[dr] & E_j \arrow[d] \\
Re_k \arrow[r] & Re_n \arrow[r] & E_k
\end{tikzcd}$$

iii,i,ii

$$\begin{tikzcd}
Re_i \arrow[d] \arrow[dr] & U \arrow[l] \arrow[d] \arrow[dr] & E_i \arrow[l] \arrow[d] \\
Re_j \arrow[r] \arrow[d] & Re_n \arrow[r] \arrow[d] \arrow[dl] & E_j \arrow[d] \arrow[dl] \\
Re_k & U \arrow[l] & E_k \arrow[l]
\end{tikzcd}$$

iii,ii,iv

$$\begin{tikzcd}
Re_i \arrow[d] & U \arrow[l] \arrow[d] \arrow[dl] & E_i \arrow[l] \arrow[d] \arrow[dl] \\
Re_j \arrow[d] \arrow[dr] & U \arrow[l] \arrow[d] \arrow[dr] & E_j \arrow[l] \arrow[d] \\
Re_k \arrow[r] & Re_n \arrow[r] & E_k
\end{tikzcd}$$

iv,iii,iii

$$\begin{tikzcd}
Re_i \arrow[d] & U \arrow[l] \arrow[d] \arrow[dl] & E_i \arrow[l] \arrow[d] \arrow[dl] \\
Re_j \arrow[d] & U \arrow[l] \arrow[d] \arrow[dl] & E_j \arrow[l] \arrow[d] \arrow[dl] \\
Re_k & U \arrow[l] & E_k \arrow[l]
\end{tikzcd}$$

iv,iv,iv

FIG. 1. Verification of multiplication in Theorem (2.4).

3. SERIAL RINGS WITH CONSTANT ADMISSIBLE SEQUENCES

An indecomposable serial ring R has a constant admissible sequence if and only if it is quasi-Frobenius (QF) [10, 16], i.e., $\mathrm{Hom}_R(\ ,{}_RR_R)$ defines a self-Morita duality. A QF ring R is weakly symmetric if the duality given by the regular bimodule ${}_RR_R$ is a weakly symmetric duality; equivalently, if $Re/Je \cong \mathrm{Soc}(Re)$ for each primitive idempotent $e \in R$ and $J = \mathrm{rad}\, R$. Even though many QF serial rings are not weakly symmetric, we shall see that all QF serial rings have a weakly symmetric duality. We will need the following characterization of an artinian ring with a weakly symmetric duality.

(3.1) PROPOSITION. *Let R be an artinian ring with $J = \mathrm{rad}\, R$. Then R has a weakly symmetric duality if and only if there is an injective cogenerator E of R-mod* *and a ring isomorphism $\varphi: R \to \mathrm{End}(_R E)$ such that $E\varphi(e) \cong E(Re/Je)$ for each primitive idempotent e of R. In particular $E = E(R/J)$; if R is basic, then E is the minimal injective cogenerator.*

Proof. Using (1.1), it remains to be shown that the duality $D = \mathrm{Hom}(\ ,{}_R E_R)$ is weakly symmetric if and only if for each primitive idempotent e of R, $Ee = E\varphi(e) \cong E(Re/Je)$. This follows from the string of equivalent statements: $Ee \cong E(Re/Je)$; $\mathrm{Soc}(Ee) \cong Re/Je$; $0 \neq e\,\mathrm{Soc}(Ee)$; $0 \neq \mathrm{Hom}(Re/Je, E)e = D(Re/Je)e$; $D(Re/Je) \cong eR/eJ$.

Of course, in order to show that R has a weakly symmetric duality, it is sufficient to show $E\varphi(e) \cong E(Re/Je)$ for each element e of a basic set of primitive idempotents of R. As an application of (3.1), we may conclude immediately from the proof of (2.4) that serial rings with strictly increasing admissible sequences have weakly symmetric dualities.

Turning our attention to serial rings with constant admissible sequences, we first remark that an indecomposable QF serial ring R with admissible sequence $c_1 = c_2 = \cdots = c_n$ is weakly symmetric if and only if $[c_j] = 1$ for some j, since $[c_j] = 1$ forces $[c_i] = 1$ and hence, $Re_i/Je_i \cong \mathrm{Soc}(Re_i)$ for all $i \in \{1,\dots, n\}$ (see (1.3)). Also note that $Re_i/Je_i \cong \mathrm{Soc}(Re_j)$ implies $Re_{[i-1]}/Je_{[i-1]} \cong \mathrm{Soc}(Re_{[j-1]})$. That every QF serial ring has a weakly symmetric duality is a consequence of

(3.2) Proposition. *Let R be a basic indecomposable QF serial ring with Kupisch series $Re_1, \dots, Re_n$. If R is not weakly symmetric, then there exists an automorphism $\psi: R \to R$ such that $\psi: e_i \mapsto e_{[i-1]}$ for $i = 1,\dots, n$.*

Proof. We may regard $R \cong \mathrm{End}(\bigoplus_{i=1}^{n} Re_i)$, so that if $r \in R$, then $r = \Sigma_{i,j} e_i r e_j$ is the sum of R-homomorphisms $e_i r e_j: Re_i \to Re_j$. For each $i \in \{1,\dots, n\}$, let $\eta_i: Re_{[i-1]} \to Je_i$ be the projective cover. We will first define ψ as a surjection from $e_i R e_j$ to $e_{[i-1]} R e_{[j-1]}$ for each pair (i, j) via $\psi: e_i r e_j \mapsto \beta_{ij}$, where β_{ij} is defined in the appropriate case below.

(i) $\mathrm{Soc}(Re_j) \cong Re_i/Je_i$. Consider the following commutative diagram:

$$\begin{tikzcd} Re_{[i-1]} \arrow[r, "\eta_i"] \arrow[d, "\beta_{ij}"'] & Re_i \arrow[d, "e_i r e_j"] \arrow[dl, dashed, "\alpha_{ij}"'] \\ Re_{[j-1]} \arrow[r, "\eta_j"] & Re_j \end{tikzcd}$$

First, note that $\mathrm{Soc}(Re_j) \cong Re_i/Je_i$ implies $i \neq j$ since R is not weakly symmetric. Hence, $Re_i r e_j \subseteq Je_j = \mathrm{im}\,\eta_j$. Therefore α_{ij} exists by the projectivity of Re_i. Also the map α_{ij} is unique, for if $\alpha_{ij}\eta_j = \alpha\eta_j$, then $e_i(\alpha_{ij} - \alpha) \in \ker \eta_j = \mathrm{Soc}(Re_{[j-1]}) \ncong Re_i/Je_i$, so $\alpha_{ij} - \alpha = 0$. Then let $\psi(e_i r e_j) = \beta_{ij} = \eta_i \alpha_{ij}$.

To show that $\psi(e_iRe_j) = e_{[i-1]}Re_{[j-1]}$, let $\beta_{ij}\colon Re_{[i-1]} \to Re_{[j-1]}$. Since $i \neq j$ in this case, $\ker \beta_{ij} \supseteq \operatorname{Soc} Re_{[i-1]} = \ker \eta_i$, so β_{ij} factors through $Re_{[i-1]}/\ker \eta_i$:

$$\begin{tikzcd} 0 \arrow[r] & Re_{[i-1]}/\ker \eta_i \arrow[r, "\bar{\eta}_i"] \arrow[d, "\beta_{ij}"] & Re_i \\ & Re_{[j-1]} & \end{tikzcd}$$

Since $Re_{[j-1]}$ is injective, there is a map $\alpha_{ij}\colon Re_i \to Re_{[j-1]}$ with $\eta_i\alpha_{ij} = \beta_{ij}$. Now let $e_ire_j = \alpha_{ij}\eta_j$.

(ii) $\operatorname{Soc}(Re_j) \not\cong Re_i/Je_i$. Consider the following commutative diagram:

$$\begin{tikzcd} Re_{[i-1]} \arrow[r, "\eta_i"] \arrow[d, dashed, "\beta_{ij}"'] \arrow[dr, dashed] & Re_i \arrow[d, "e_ire_j"] \arrow[d, phantom] \\ Re_{[j-1]} \arrow[r, "\eta_j"] & Re_j \end{tikzcd}$$

$$\alpha_{ij}$$

Let $\alpha_{ij} = \eta_ie_ire_j$. Now $\operatorname{im} \alpha_{ij} \subseteq Je_j = \operatorname{im} \eta_j$, so β_{ij} exists since $Re_{[i-1]}$ is projective. And β_{ij} is unique, for if $\beta_{ij}\eta_j = \beta\eta_j$, then $e_{[i-1]}(\beta_{ij} - \beta) \in \ker \eta_j = \operatorname{Soc}(Re_{[j-1]}) \not\cong Re_{[i-1]}/Je_{[i-1]}$, so $\beta_{ij} - \beta = 0$.

To see that $\psi(e_iRe_j) = e_{[i-1]}Re_{[j-1]}$, let $\beta_{ij}\colon Re_{[i-1]} \to Re_{[j-1]}$ be given and set $\alpha_{ij} = \beta_{ij}\eta_j$. Then $Re_{[i-1]}\alpha_{ij} \subseteq Je_j$, so $\ker \eta_i = \operatorname{Soc}(Re_{[i-1]}) \subseteq \ker \alpha_{ij}$, and thus α_{ij} factors through $Re_{[i-1]}/\ker \eta_i$:

$$\begin{tikzcd} 0 \arrow[r] & Re_{[i-1]}/\ker \eta_i \arrow[r, "\bar{\eta}_i"] \arrow[d, "\bar{\alpha}_{ij}"] & Re_i \\ & Re_j & \end{tikzcd}$$

Since Re_j is injective, there is a map $e_ire_j\colon Re_i \to Re_j$ with $\eta_ie_ire_j = \alpha_{ij}$ and $\psi(e_ire_j) = \beta_{ij}$.

Define a surjection $\psi\colon R \to R$ via $\psi\colon r = \Sigma_{i,j}e_ire_j \mapsto \Sigma_{i,j}\beta_{ij}$. A simple argument shows that $\beta_{ij} + \beta'_{ij}$ is the map associated with $e_ire_j + e_ir'e_j$, and it follows that ψ is additive. To show that ψ is multiplicative, it is sufficient to show that for all $i, j, k \in \{1,\ldots, n\}$, $\psi(e_ire_jr'e_k) = \beta_{ij}\beta'_{jk}$. There are eight different diagrams to check, exhibited in Fig. 2, depending on the pairs (i, j), (j, k), and (i, k). The only combination for which there is any difficulty in verifying that $\beta_{ij}\beta'_{jk}$ corresponds to $e_ire_jr'e_k$ is $\operatorname{Soc}(Re_j) \not\cong Re_i/Je_i$, $\operatorname{Soc}(Re_k) \not\cong Re_j/Je_j$, $\operatorname{Soc}(Re_k) \cong Re_i/Je_i$. We will demonstrate this case. Assume that we are given a commutative diagram

$$\begin{tikzcd}
Re_{[i-1]} \arrow[r, "\eta_i"] \arrow[d, "\beta_{ij}"'] \arrow[dr, "\alpha_{ij}"] & Re_i \arrow[d, "e_i r e_j"] \\
Re_{[j-1]} \arrow[r, "\eta_j"] \arrow[d, "\beta'_{jk}"'] \arrow[dr, "\alpha_{jk}"] & Re_j \arrow[d, "e_j r' e_k"] \\
Re_{[k-1]} \arrow[r, "\eta_k"] & Re_k .
\end{tikzcd}$$

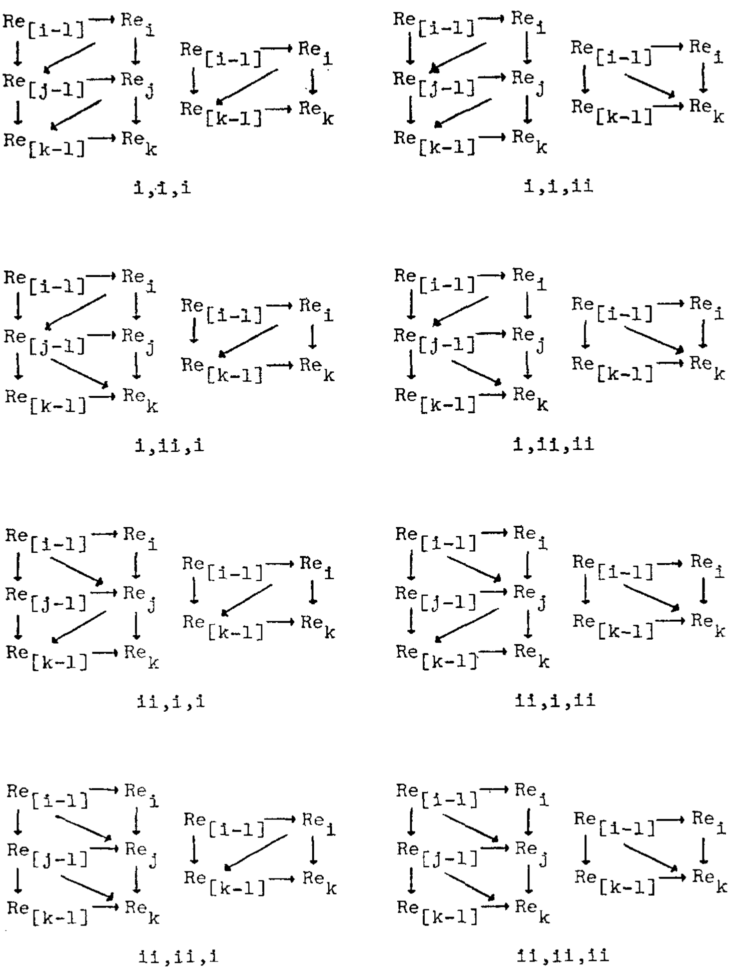

FIG. 2. Verification of multiplication in Proposition (3.2).

We need to define a map $\alpha_{ik}: Re_i \to Re_{[k-1]}$ such that $\alpha_{ik}\eta_k = e_ire_jr'e_k$. Accordingly, let $x \in Re_i$. Since under our present hypotheses $i \neq j$ so that $\operatorname{im} e_ire_j \subseteq Je_j$, there exists $y \in Re_{[j-1]}$ such that $y\eta_j = xe_ire_j$. Define α_{ik}: $Re_i \to Re_{[k-1]}$ via $\alpha_{ik}: x \mapsto y\beta'_{jk}$. Then α is an R-homomorphism and is well-defined since our present hypotheses imply $j \neq k$, so that $\ker \beta'_{jk} \supseteq \operatorname{Soc}(Re_{[j-1]}) = \ker \eta_j$. Routine diagram chasing shows that $\alpha_{ik}\eta_k = e_ire_jr'e_k$ and $\eta_i\alpha_{ik} = \beta_{ij}\beta'_{jk}$. Therefore the following diagram is commutative, showing that $\psi(e_ire_jr'e_k) = \beta_{ij}\beta'_{jk}$:

$$\begin{tikzcd}
Re_{[i-1]} \arrow[r, "\eta_i"] \arrow[d, "\beta_{ij}\beta'_{jk}"'] & Re_i \arrow[d, "e_ire_jr'e_k"] \arrow[dl, "\alpha_{ik}"'] \\
Re_{[k-1]} \arrow[r, "\eta_k"'] & Re_k .
\end{tikzcd}$$

Because R is artinian and $\psi: R \to R$ is a surjective ring homomorphism, ψ is also an isomorphism. Finally, $\psi: e_i \mapsto e_{[i-1]}$ since the diagram

$$\begin{tikzcd}
Re_{[i-1]} \arrow[r, "\eta_i"] \arrow[d, "e_{[i-1]}"'] \arrow[dr, "\eta_ie_i"'] & Re_i \arrow[d, "e_i"] \\
Re_{[i-1]} \arrow[r, "\eta_i"'] & Re_i
\end{tikzcd}$$

is commutative, and the proof is complete.

Suppose that R is a basic indecomposable QF serial ring that is not weakly symmetric, fix $j \in \{1,..., n\}$, and let ψ be the automorphism of (3.2). Then as k varies from 1 to n, $\psi^k(e_j) = e_{[j-k]}$ runs through $\{e_1, ..., e_n\}$, and every simple left module appears once in the set $\{\operatorname{Soc}(Re_{[j-k]})\}_{k=1}^n$. To each ψ^k corresponds a bimodule ${}_RM_{k\,R}$ with the left module ${}_RM_k = {}_RR$ and sclar multiplication on the right given by $m \cdot r = m\psi^k(r)$ for $m \in M_k$ and $r \in R$. Then $D_k = \operatorname{Hom}(\ , {}_RM_{k\,R})$ is a Morita duality between R-mod and mod-R with $D_k(\operatorname{Soc}(Re_{[j-k]})) \cong e_jR/e_jJ$; in particular, there is a k with $D_k(Re_j/Je_j) \cong e_jR/e_jJ$. By (1.4) this value of k is $[1 - c(e_jR_R)]$. Since $c(e_jR_R) = c(e_iR_R)$ for all $i \in \{1,..., n\}$, (1.4) also shows that $E(Re_i/Je_i) \cong Re_{[i-k]} = R\psi^k(e_i)$ for any i. Applying (3.1), this proves:

(3.3) THEOREM. *Every QF serial ring has a weakly symmetric duality.*

(3.4) EXAMPLE. Basic indecomposable weakly symmetric QF serial rings exist for which (3.2) fails. To construct such a ring, let $\Lambda = \mathbb{Z}_2[x]/(x^2)$. Define functions $\alpha: \mathbb{Z}_2 \to \mathbb{Z}_4$ and $\beta: \mathbb{Z}_2 \to \Lambda$ via $\alpha(0) = 0$, $\alpha(1) = 2$ and $\beta(0) = 0$, $\beta(1) = x + (x^2)$. Notice that α is a homomorphism of $\mathbb{Z}_4$-modules, β is a homomorphism of Λ-modules, and for $y, z \in \mathbb{Z}_2$, $y \cdot \alpha(z) = \alpha(z) \cdot y = 0 = \beta(z) \cdot y = y \cdot \beta(z)$. Let R be the set of all (2×2)-matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $a \in \mathbb{Z}_4$; $b, c \in \mathbb{Z}_2$; and $d \in \Lambda$. Define addition as usual and multiplication by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} aa' + \alpha(bc') & ab' + bd' \\ ca' + dc' & \beta(cb') + dd' \end{pmatrix},$$

where the products ab' and bd' are computed in ${}_{\mathbb{Z}_4}\mathbb{Z}_{2\Lambda}$ ca' and dc' in ${}_{\Lambda}\mathbb{Z}_{2\mathbb{Z}_4}$, aa' in $\mathbb{Z}_4$, dd' in Λ, and bc' and cb' in $\mathbb{Z}_2$. Then R is a ring with radical J consisting of matrices with $a \in \operatorname{rad} \mathbb{Z}_4$; $b, c \in \mathbb{Z}_2$; and $d \in \operatorname{rad} \Lambda$. Let $e_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $e_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Simple computation checks that R is a basic indecomposable serial ring with admissible sequence 3,3; hence, R is QF and weakly symmetric. Any ring automorphism $\psi: R \to R$ with $\psi(e_1) = e_2$ would induce an isomorphism between $e_1Re_1 \simeq \mathbb{Z}_4$ and $e_2Re_2 \simeq \Lambda$, a contradiction. We arrive at the same contradiction if we assume the existence of a duality D with $D(Re_1/Je_1) \simeq e_2R/e_2J$.

4. Factor Rings and Good Dualities

A question closely related to "What rings have self-duality?" is "Do factor rings of rings with self-duality have self-duality?" It is well-known that if there is a Morita duality between R-mod and mod-S given by $\operatorname{Hom}(\ , {}_RU_S)$ and I is any ideal of R, then $\operatorname{Hom}(\ , {}_{R/I}U'_{S/H})$ defines a duality between R/I-mod and mod-S/H, where $U' = r_U(I)$ and $H = r_S(U')$ [13]. Roux [21], at the suggestion of B. J. Müller, defines a self-duality for R induced by $\varphi: R \to S$ to be *good* if for every ideal I of R, $\varphi(I) = r_S r_U(I)$. Hence, if R has a good duality, then every factor ring of R has self-duality. It is not hard to see that a good duality is weakly symmetric. The next proposition gives a sufficient condition for the converse to hold. An R-module M is said to be *distributive* if its lattice of submodules is a distributive lattice.

(4.1) Proposition. *Let R be an artinian ring. If every primitive right (or left) ideal of R is distributive, then every weakly symmetric duality for R is a good duality.*

Proof. Let $\{e_1, \ldots, e_n\}$ be a complete set of primitive idempotents for R. By (3.1) we may choose an injective cogenerator E of R-mod and a ring isomorphism $\Phi: R \to \operatorname{End}({}_RE) = S$ that induces the weakly symmetric duality

$D = \operatorname{Hom}_R(\ , {}_RE_R)$ with $E\Phi(e_i) = E(Re_i/Je_i)$. Let $f_i = \Phi(e_i)$ and $E_i = Ef_i$. Also set $E' = r_E(I)$, $E_i' = r_{E_i}(I) = r_E(I)f_i$, and $H = r_S(E')$. We need to show that $H = \Phi(I)$. First, notice that $f_iH_S \simeq \operatorname{Hom}_R(E_i/E_i', E) = D(E_i/E_i')$ and $D(Re_j/Je_j)_S \simeq f_jS/f_j(\text{rad } S)$. Since e_iR is distributive for each i, also $D(e_iR) \simeq {}_RE_i$ and hence ${}_RE_i'$ are distributive. Therefore [6, Theorem 2.4; 7, Lemma 4], $c({}_{e_jRe_j}e_jE_i') = c(e_i(R/I)\, e_{je_jRe_j})$. Thus,

$$\begin{aligned} c(f_iHf_{jf_jSf_j}) &= c({}_{e_jRe_j}e_j(E_i/E_i')) = c({}_{e_jRe_j}e_jE_i) - c({}_{e_jRe_j}e_jE_i') \\ &= c(e_iRe_{je_jRe_j}) - c(e_i(R/I)e_{je_jRe_j}) = c(e_iIe_{je_jRe_j}). \end{aligned}$$

Because f_iS is distributive, $f_iSf_{jf_jSf_j}$ is uniserial [7, Lemma 4]. Both f_iHf_j and $\Phi(e_iIe_j) = f_i\Phi(I)f_j$ are submodules of $f_iSf_{j\ f_jSf_j}$ and have the same composition length, so $f_iHf_j = f_i\Phi(I)f_j$. Thus $H = \Phi(I)$ and D is a good duality.

(4.2) Example. To construct an example of a weakly symmetric duality that is not a good duality, let Δ be a division ring and R' the subring of the (8×8)-matrix ring over Δ with typical element

$$\begin{bmatrix} a & d & x & s & & & & \\ & a & c & y & & \bigcirc & & \\ & & a & b & & & & \\ & & & a & & & & \\ & & & & a & c & y & t \\ & \bigcirc & & & & a & b & z \\ & & & & & & a & d \\ & & & & & & & a \end{bmatrix}.$$

Let $K \subseteq R'$ be the ideal given by the conditions $a = b = c = d = 0$ and $x + y + z = 0$. Let $R = R'/K$. Then $\operatorname{Soc}(R)$ is simple, so R is a local QF ring with a weakly symmetric duality $\operatorname{Hom}_R(\ , {}_RR_R)$. Let $I \subseteq R$ be the ideal given by $a = b = c = 0$. Then $H = r_Rr_R(I)$ is the ideal given by $a = c = d = 0$, so $H \neq I$ and the duality $\operatorname{Hom}(\ , {}_RR_R)$ is not good. It should be noted that R/I nevertheless has self-duality, for $\psi' \colon R' \to R'$ via

$$\psi' \colon \begin{bmatrix} a & d & x & s & & & & \\ & a & c & y & & \bigcirc & & \\ & & a & b & & & & \\ & & & a & & & & \\ & & & & a & c & y & t \\ & \bigcirc & & & & a & b & z \\ & & & & & & a & d \\ & & & & & & & a \end{bmatrix} \mapsto \begin{bmatrix} a & c & y & t & & & & \\ & a & b & z & & \bigcirc & & \\ & & a & d & & & & \\ & & & a & & & & \\ & & & & a & d & x & s \\ & \bigcirc & & & & a & c & y \\ & & & & & & a & b \\ & & & & & & & a \end{bmatrix}$$

induces an automorphism ψ of R with $\psi(I) = H$.

In contraposition to the ring R of (4.2) we now show

(4.3) PROPOSITION. *Let R be an indecomposable serial ring with strictly increasing admissible sequence. Every self-duality D for R is a good duality.*

Proof. By (4.1) it is sufficient to show that every self-duality D for R is weakly symmetric. Let $\{e_1, ..., e_n\}$ be a basic set of primitive idempotents for R. Since Re_i is the projective cover of Re_i/Je_i, $D(Re_i) \cong E(D(Re_i/Je_i))$ and $c({}_RRe_i) = c(E(D(Re_i/Je_i))_R)$. (See [1, Sects. 23 and 24].) But by (1.4) $c({}_RRe_i) = c(E(e_iR/e_iJ))$. Because the admissible sequence of R is strictly increasing, there is at most one indecomposable projective left R-module of a given length, and thus at most one indecomposable injective right R-module of a given length. Therefore $E(e_iR/e_iJ) \cong E(D(Re_i/Je_i))$ and $e_iR/e_iJ \cong D(Re_i/Je_i)$.

Murase has shown that many serial rings are isomorphic to factors of QF serial rings [17, Theorem 9], including all serial rings R with the nilpotency index of rad R less than or equal to the cardinality of a basic set of primitive idempotents for R. As immediate consequences of (2.4), (3.3), (4.1), (4.3), and [17, Theorem 9], we have:

(4.4) THEOREM. *Every factor ring of a serial ring with either a strictly increasing or constant admissible sequence has self-duality.*

(4.5) COROLLARY. *Let R be a serial ring with a basic set of primitive idempotents $\{e_1, ..., e_n\}$. If* $(\operatorname{rad} R)^n = 0$, *then R has self-duality.*

Were it true that all serial rings are factors of serial rings with either a strictly increasing or constant admissible sequence, we could conclude that all serial rings have self-duality. This hope must be abandoned as the next example shows. In the discussion of the example we will use a result due to Ivanov [9, Theorem 11] stating that:

(4.6) *If e and f are primitive idempotents in an indecomposable serial ring R with $J =$* rad R *and $c = c({}_{eRe}eRf)$ then*

(1) $c = c(eRf_{fRf})$, *and*

(2) $eRe/(eJe)^c \cong fRf/(fJf)^c$.

(4.7) EXAMPLE. Let Λ, α, and β be defined as in (3.4). Let R consist of (3×3)-matrices of the form

$$\begin{pmatrix} a & s & t \\ u & b & d \\ v & w & c \end{pmatrix}$$

with $a \in \mathbb{Z}_4$; $s, t, u, v, w \in \mathbb{Z}_2$; and $b, c, d \in \Lambda$. Define addition as usual and multiplication by

$$\begin{pmatrix} a & s & t \\ u & b & d \\ v & w & c \end{pmatrix}\begin{pmatrix} a' & s' & t' \\ u' & b' & d' \\ v' & w' & c' \end{pmatrix}$$

$$= \begin{pmatrix} aa' + \alpha(su') + \alpha(tv') & as' + sb' & at' + sd' + tc' \\ ua' + bu' + dv' & \beta(us') + bb' + \beta(dw') & \beta(ut') + bd' + dc' \\ va' + cv' & vs' + wb' + cw' & \beta(vt') + \beta(wd') + cc' \end{pmatrix},$$

where the products su', tv', us', ut', vs', and vt' are computed in $\mathbb{Z}_2$; as', at', sb', sd', tc', wb', and wd' in ${}_{\mathbb{Z}_4}\mathbb{Z}_{2\Lambda}$; bu', cv', cw', dv', dw', ua', and va' in ${}_{\Lambda}\mathbb{Z}_{2\mathbb{Z}_4}$; aa' in $\mathbb{Z}_4$; and bb', bd', cc', and dc' in Λ.Setting

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

the corresponding admissible sequence of R is 4, 4, 5. If R were a factor of a serial ring with a strictly increasing admissible sequence, then R would be a factor of a serial ring R' with admissible sequence 4, 5, 6. But by (1.3) and (4.6), if e and f are primitive idempotents of R', then $c({}_{eR'e}eR'f) = 2$, $eR'e \cong fR'f$, and we arrive at a contradiction $\mathbb{Z}_4 \cong e_1Re_1 \cong e_2Re_2 \cong \Lambda$. Similarly, if R were a factor of a serial ring with a constant admissible sequence, then R would be a factor of a serial ring R' with admissible sequence 5, 5, 5, and (1.3) and (4.6) give us the same contradiction.

Even though the ring R of (4.7) fails to be a factor of a serial ring with a strictly increasing or constant admissible sequence, it is still true that R has self-duality. In fact, let R be any serial ring with admissible sequence 4, 4, 5. Then by considering the maps $Re_1 = E(Re_1/Je_1)$, $0 \to Re_2 \to E(Re_2/Je_2)$, and $Re_3 \to E(Re_3/Je_3) \to 0$ and using the techniques in the proofs of (2.4) and (3.2), one can show that R has self-duality. Similar computations prove that all serial rings with a basic set of at most three idempotents have self-duality; unfortunately, each class of serial rings with admissible sequences equivalent modulo 3 requires different diagrams. We know of no example of a serial ring without self-duality.

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Note added in proof. R. B. Warfield has informed us of a shorter proof of (2.4) by noting that serial rings with strictly increasing admissible sequences are factor rings of (S, M)-upper triangular matrix rings, showing that such rings have self-duality, and then applying (4.1).

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